Complete Noncompact Kähler Manifolds with Positive Bisectional Curvature

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§0 Preliminary

Let M^n be a complex n-dimensional Kähler manifold with a Kähler metric

$$ds^2 = \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta.$$

We have the following terminologies for its curvatures:

Curvature tensor:

$$R_{\alpha\bar{\beta}\mu\bar{\nu}} = -\frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial z^\mu \partial \bar{z}^\nu} + \sum_{\lambda,\sigma=1}^{n} g^{\lambda\bar{\sigma}} \frac{\partial g_{\lambda\bar{\beta}}}{\partial z^\mu} \frac{\partial g_{\alpha\bar{\sigma}}}{\partial \bar{z}^\nu}$$

$$\Big(= R_m(\tfrac{\partial}{\partial z^\alpha}, \tfrac{\partial}{\partial \bar{z}^\beta}, \tfrac{\partial}{\partial z^\mu}, \tfrac{\partial}{\partial \bar{z}^\nu}) \Big)$$

Ricci curvature:

$$R_{\alpha\bar{\beta}} = \sum_{\mu,\nu=1}^{n} g^{\mu\bar{\nu}} R_{\alpha\bar{\beta}\mu\bar{\nu}} = -\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} (\log \det g_{\mu\bar{\nu}})$$

$$\Big(= Ric(\tfrac{\partial}{\partial z^\alpha}, \tfrac{\partial}{\partial \bar{z}^\beta}) \Big)$$

Scalar curvature:

$$R = \sum_{\alpha,\beta=1}^{n} g^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$$

Bisectional curvature:

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}}\Big(= R_m(\tfrac{\partial}{\partial z^\alpha}, \tfrac{\partial}{\partial \bar{z}^\alpha}, \tfrac{\partial}{\partial z^\beta}, \tfrac{\partial}{\partial \bar{z}^\beta})\Big)$$

Choose an orthonormal basis $e_1, \cdots, e_{2n}$ with $Je_\alpha = e_{n+\alpha}$ for $\alpha = 1, \cdots, n$, where J is the compatible almost complex structure of M^n.

Set $u_\alpha = \frac{1}{\sqrt{2}}(e_\alpha - \sqrt{-1}Je_\alpha)$, $\alpha = 1, \cdots, n$ then $\{u_\alpha\}$ is a unitary basis.

- The bisectional curvature is

$$R_m(u_\alpha, \bar{u}_\alpha, u_\beta, \bar{u}_\beta) = R_m(e_\alpha, e_\beta, e_\beta, e_\alpha) + R_m(e_\alpha, Je_\beta, Je_\beta, e_\alpha)$$

as the sum of two Riemannian sectional curvatures.

- The Ricci curvature is

$$Ric(u_\alpha, \bar{u}_\alpha) = \sum_{\beta=1}^{n} R_m(u_\alpha, \bar{u}_\alpha, u_\beta, \bar{u}_\beta).$$

§1 Motivation

The classical uniformization theorem says that a simply connected Riemann surface is biholomorphic to $\mathbb{S}^2$ (Riemann sphere), or $\mathbb{C}$, or D (unit disc).

• It gives the characterization for the standard complex structures of one-dimensional K$\ddot{a}$hler manifolds.

• There is a vast variety of biholomorphically distinct complex structures on $\mathbb{R}^{2n}$ for $n > 1$.

Thus, in order to characterize the standard complex structures for higher dimensional K$\ddot{a}$hler manifolds, one must *impose more restrictions* on the manifolds.

From the point of view of differential geometry, one consequence of uniformization theorem is that

A positively curved compact or noncompact Riemann surface must be biholomorphic to the Riemann sphere $\mathbb{S}^2$ ***or the complex line*** $\mathbb{C}$ ***respectively***.

Naturally, one would ask whether there is similar characterization for higher dimensional complete K$\ddot{a}$hler manifolds with positive "curvature".

Higher dimensional uniformization problems

• **Frankel conjecture**:

Kähler manifold M^n, compact, bisect> 0

$$\Longrightarrow M^n \overset{biholo.}{\cong} \mathbb{C}P^n.$$

• **Yau conjecture**:

Kähler manifold M^n, complete noncompact, bisect> 0

$$\Longrightarrow M^n \overset{biholo.}{\cong} \mathbb{C}^n.$$

• **A weaker version (Greene-Wu-Yau)**:

Kähler manifold M^n, complete noncompact, sect> 0

$$\Longrightarrow M^n \overset{biholo.}{\cong} \mathbb{C}^n.$$

The Frankel conjecture was completely resolved by Mori, Siu-Yau in 1979. And the generalized Frankel conjecture (with bisect≥ 0) was also completely resolved by Mok in 1986. Thus in the rest of this talk, we will take the attention to *Yau conjecture or its weak version for complete noncompact Kähler manifolds.*

§2 Geometric Properties

Consider complex n-dimensional Kähler manifold $(M^n, g_{\alpha\bar{\beta}})$ with nonnegative bisectional curvature.

Volume Growth

Recall: for a (real) m-dimensional Riemannian manifold M^m with $Ric(M^m) \geq 0$

• (**Bishop**) $Vol(B(x_0, r)) \leq \omega_m r^m, \quad \forall r \geq 0$
where ω_m is the volume of unit ball of $\mathbb{R}^m$.

• (**Calabi, Yau**) $Vol(B(x_0, r)) \geq c(x_0) r, \quad \forall r \geq 1$
where $c(x_0)$ is a positive constant.

Example: Let $M^m = X \times \mathbb{R}$, where X is a compact Riemannian manifold with $Ric(X) \geq 0$. Then

$$Vol(B(x_0, r)) \leq Const. \cdot r, \qquad as \;\; r \;\; large.$$

This shows that the Calabi-Yau's lower bound estimate is *sharp*.

Proposition (Chen-Zhu, *QJPAM* (2005))

Complex n-dimensional Kähler manifold M^n, bisect≥ 0 everywhere, and bisect> 0 at least one point

$$\Longrightarrow Vol(B(x_0, r)) \geq c(x_0) r^n, \quad \forall r \geq 1,$$

where $c(x_0)$ is a positive constant.

Remarks

(1) *The assumption that "bisect>0 at least one point" is necessary.*
e.g.
$M_1^{n-1} = \mathbb{C}P^{n-1}, \text{and} M_2 = S^1 \times \mathbb{R}$, then $M^n = M_1^{n-1} \times M_2$ has nonnegative bisectional curvature and its volume growth satisfies

$$Vol(B(x_0, r)) \leq const. \cdot r, \quad as \;\; r \;\; large.$$

(2) *One can not expect "Riemannian manifold M^m, sect$> 0 \Longrightarrow Vol(B(x_0, r)) \geq c(x_0) r^{\frac{m}{2}}, \quad \forall r \geq 1$".*

(3) *Klemback, Cao constructed some complete Kähler metrics on* $\mathbb{C}^n$ which have

$$\begin{cases} \text{positive bisectional curvature} & \\ Vol(B(x_0, r)) = const. \cdot r^n, & \text{as } r \to +\infty \\ |R_m(x)| = O(\frac{1}{r}), & \text{as } r = d(x, x_0) \to +\infty \end{cases}$$

In particular, this shows that our volume estimate is *sharp.*

Curvature decay

Recall: Bonnet-Myres theorem says that

$$\text{Riemannian manifold } M^m, Ric(M^m) \geq \delta > 0$$

$$\Rightarrow M^m \text{ is compact}$$

In other words: *Any complete noncompact Riemaniann manifold M^m with $Ric(M^m) \geq 0$ must have*

$$\inf\{|Ric(x)| \mid x \in B(x_0, r)\} \to 0, \ \textit{as } r \to +\infty$$

The following proposition gives a *quantitative* version of Bonnet-Myres theorem.

Proposition (Chen-Zhu, *QJPAM*(2005))

$$\text{Kähler manifold } M^n, \text{ bisect} > 0$$

$$\Rightarrow \frac{1}{Vol(B(x_0, r))} \int_{B(x_0,r)} R(x)dx \leq \frac{C(x_0)}{1+r}, \quad \forall r \geq 0.$$

Remarks.

(1) The above Klemback-Cao examples show that this linear decay estimate for curvature is *sharp*.

(2) The assumption that "bisect> 0" is somewhat necessary, e.g. $\mathbb{C}P^k \times \mathbb{C}^{n-k}$ has (positive) constant scalar curvature.

§3 Approach via Elliptic Equations

The first result for the uniformization conjecture is the following isometrically embedding theorem.

Theorem (Mok-Siu-Yau, *Compositio Math.*(1981))

Kähler manifold M^n, $n \geq 2$, complete noncompact, bisect$\geq$0 and

(i) $Vol(B(x_0, r)) \geq C_1 r^{2n}, \quad \forall r \geq 0$

(ii) $R(x) \leq \frac{C_2}{(1+d(x,x_0))^{2+\varepsilon}}, \quad$ *on* M^n, *(*$\varepsilon > 0$*)*

$$\Rightarrow M^n \underset{biholo.}{\overset{isom.}{\cong}} \mathbb{C}^n \text{ with the flat metric.}$$

Ideas of proof

Try to solve the Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar{\partial}u = Ric \text{ on } M^n.$$

This is an *overdeterminate* system. They first considered the following Poisson equation

$$\triangle u = R \text{ on } M^n.$$

• Since the Green function $G(x, x_0) \approx \frac{d(x,x_0)^2}{Vol(B(x_0,d(x,x_0)))} \approx Const.d(x, x_0)^{2-2n}$

the fasten quadratic curvature decay $|R(x)| = O(\frac{1}{d(x,x_0)^{2+\varepsilon}})$

$$\Rightarrow u(x) \approx \int G(y,x)R(y)dy \text{———— bounded}$$

• (Bochner type trick)

$$\triangle \|\sqrt{-1}\partial\bar{\partial}u - Ric\|^2 \geq 0$$

$\Rightarrow$ the function u solve the Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar{\partial}u = Ric \geq 0$$

• Establish a Liouville type theorem for the bounded plurisubharmonic function

$$\text{i.e. } u(x) \equiv Const. \text{ on } M^n$$

Roughly says:

• use the function u as weight in L^2-estimates of $\bar{\partial}$-operator to get a holomorphic function f which is nontrivial, bounded and $\int |f|e^{-u} < +\infty$,
• By Yau's Liouville theorem, one has $f \equiv const.$
A contradiction!

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From the above argument, one can see:

$$\begin{cases} R \le \frac{C}{(1+d(x,x_0))^{2+\varepsilon}} & \Rightarrow \quad u \text{ is bounded,} \\ R \le \frac{C}{(1+d(x,x_0))^{2}} & \Rightarrow \quad u \text{ is of logarithmic growth.} \end{cases}$$

Thus in the quadratic decay case "$R \le \frac{C}{(1+d(x,x_0))^2}$" one can use the solution u of the Poincaré-Lelong equation as weight in L^2-theory of $\bar{\partial}$-operator to obtain a holomorphic function of *polynomial growth.*

Theorem (Mok, *Bull Soc. Math. France* (1984))

Kähler manifold M^n*, complete noncompact, bisect*>0 *and*

(i) $Vol(B(x_0,r)) \ge C_1 r^{2n}, \quad \forall r \ge 0,$

(ii)' $R(x) \le \frac{C_2}{(1+d(x,x_0))^2}$*, on* M^n

$$\Rightarrow M^n \overset{biholo.}{\cong} \text{ an affine algebraic variety.}$$

Moreover if $n = 2$ *and*

(iii) sect$(M^n) > 0$

$$\Rightarrow M^2 \overset{biholo.}{\cong} \mathbb{C}^2$$

Ideas of proof

• Use the quadratic curvature decay assumption to solve the Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar{\partial}u = Ric$$

to get a strictly plurisubharmonic function of *logarithmic growth*.

• By the L^2-theory of $\bar{\partial}$-operator, one can use the function u as weight to get algebraically independent holomorphic functions $f_1, \cdots, f_n$ of polynomial growth.

Denote by

$$P(M^n) = \text{ the algebra of holomorphic functions of}$$
$$\text{polynomial growth on } M^n,$$
$$R(M^n) = \text{ the quotient field of } P(M^n).$$

• Get a multiplicity estimate for the zero of a holomorphic function of polynomial growth. Then by a Poincare-Siegel type argument,

$$R(M^n) = \text{ a finite extension field of } \mathbb{C}(f_1, \cdots, f_n)$$
$$= \mathbb{C}(f_1, \cdots, f_n, g/h)$$

(by primitive element theorem, for some $g, h \in P(M^n)$).

• The map $F = (f_1, \cdots, f_n, g, h)$ defines, in an appropriate sense, a birational equivalence between M^n and some irreducible affine algebraic subvariety of $\mathbb{C}^{n+2}$.

• Establish uniform estimates on multiplicity and the number of irreducible components of the zero sets of a holomorphic function or several holomorphic functions to desingularize the map F.

• In the case "$n = 2$ and $\mathrm{Sect}(M^n) > 0$", Mok used a theorem of Ramanujam which states that "an algebraic variety of homeomorphic to $\mathbb{R}^4$ is biholomorphic to $\mathbb{C}^2$".

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In views of the above general compactifying scheme of Mok, one needs to overcome the following two difficulties to obtain an answer for the uniformization conjecture:

Question 1 *How to get the topology of M^n under the assumption "bisect> 0"?*

• Cheeger-Gromoll-Meyer(1969, 1971):

$$\text{Riemannian manifold } M^m, \ \mathrm{sect}(M^m) > 0$$
$$\Rightarrow M^m \text{ is diffeomorphic to } \mathbb{R}^m.$$

The main tool is the Topogonov triangle comparison theorem which holds under the *sectional* curvature assumption.

• Do not know whether the Topogonov triangle comparison still holds under the holomorphic *bisectional* assumption.

Question2 *Can one remove the curvature decay assumption?*

• Mok-Siu-Yau assumed that the *faster than quadratic* curvature decay assumption

$$R(x) \leq \frac{C_2}{(1+d(x,x_0))^{2+\varepsilon}}$$

• Mok assumed that the *quadratic* curvature decay assumption

$$R(x) \leq \frac{C_2}{(1+d(x,x_0))^{2}}$$

• Chen-Zhu showed that the *linear* curvature decay always holds in average sense

$$\frac{1}{Vol(B(x_0,r))} \int_{B(x_0,r)} R(x)dV \leq \frac{C_2}{(1+r)}$$

§4 **Parabolic Equations**

Let M^n be a complex n-dimensional K$\ddot{a}$hler manifold with K$\ddot{a}$hler metric $g_{\alpha\bar{\beta}}(x)$.

The **Ricci flow** is the following evolution equations

$$\begin{cases} \frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x,t) = -R_{\alpha\bar{\beta}}(x,t), & \text{on } M^n \times [0,T) \\ g_{\alpha\bar{\beta}}(x,0) = g_{\alpha\bar{\beta}}(x), & \text{on } M^n. \end{cases}$$

Note that $-R_{\alpha\bar{\beta}}(x,t) = \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}(\log\det(g_{\mu\bar{\nu}}))$. Thus the Ricci flow is a *parabolic* system (of nonlinear Monge-Ampere type).

• **Short time existence** (Hamilton, Shi):

If the curvature of the initial metric is bounded, then the solution of the Ricci flow exists on a short time interval.

• **Preserving K$\ddot{a}$hlerity** (Hamilton, Shi):

$g_{\alpha\bar{\beta}}(x,0)$ is K$\ddot{a}$hler $\Rightarrow$ $g_{\alpha\bar{\beta}}(x,t)$ is also K$\ddot{a}$hler for each $t > 0$

• **Preserving positive bisectional curvature** (Bando, Mok, Shi):

bisect>0 at $t=0 \Rightarrow$ bisect>0 for each $t>0$

Thus to study the topological and complex structure of a complete noncompact Kähler manifold of positive holomorphic bisectional curvature, we can

replace the Kähler metric by any one of the evolving metric of the Ricci flow.

In particular, if we can get the long time behaviors of the solution of the Ricci flow, it will be possible to extract informations to determine the topological and complex structure of the given Kähler manifold.

Shi is the first one to use the Ricci flow to approach the Yau's conjecture. It is pity that his arguments have several gaps.

§5 Approach via Parabolic Equations

Theorem (Chen-Tang-Zhu, *J. Diff. Geom.* (2004))

Suppose: Kähler surface M^2, $0 <$*bisect*$\leq$ *Const.*, *and*

(i) $Vol(B(x_0, r)) \geq C_1 r^4, \quad \forall r \geq 0.$

Then

$$M^2 \overset{biholo.}{\cong} \mathbb{C}^2.$$

Ideas of proof

Step 1. **Long Time Behaviors**.

Study the *long time behaviors* of the solution of the Ricci flow

$$\begin{cases} \frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x,t) = -2R_{\alpha\bar{\beta}}(x,t), & \text{on } M^2 \times [0,T) \\ g_{\alpha\bar{\beta}}(x,0) = \text{ the given Kähler metric.} \end{cases}$$

Lemma (Preserving maximal volume growth)

Under the assumption of the theorem, we still have

$$Vol_t(B_t(x,r)) \geq C_1 r^4,$$

for all $r > 0$, $x \in M^2$ *and* $t \in [0,T)$, *where* $B_t(x,r)$*is the geodesic ball of radius* r *centered at* x *and* Vol_t *is* *the volume taken w.r.t.* $g_{\alpha\bar{\beta}}(\cdot, t)$.

Let T be the maximal time (i.e., if $T < +\infty$, then the curvature $|R_m(\cdot, t)|$ become unbounded as $t \to T$). According to Hamilton, we classify the solution into following types:

Type I: $T < +\infty$ and $\sup(T - t)|R_m(x, t)| < +\infty$
Type II(a): $T < +\infty$ and $\sup(T - t)|R_m(x, t)| = +\infty$
Type II(b): $T = +\infty$ and $\sup t|R_m(x, t)| = +\infty$
Type III: $T = +\infty$ and $\sup t|R_m(x, t)| < +\infty$

To understand the structure of the solution near the maximal time T, we rescale the solution around a sequence points $P_j \in M^2$ and a sequence of time $t_j \to T$. Then any Type I, or Type II(a),(b) limit must be an ancient solution (i.e. the solution exists on the time interval $(-\infty, 0]$) with $0 \leq$bisect$\leq Const.$).

By using the dimension reduction argument of Hamilton and a trick of Ivey, we obtained

Lemma: *Any complex two-dimensional ancient solution, with* $0 \leq$*bisect*$\leq Const.$, *must have*

$$\nu = \lim_{r\to+\infty} \frac{Vol_t(B_t(x_0,r))}{r^4} = 0,$$

for each t.

Remarks:

• Independently, Perelman also obtained this result for the Ricci flow on Riemannian manifold with nonnegative curvature operator.

• Most recently, Ni generalized this lemma for all dimensions by the combination of an idea of Perelman and the above linear decay estimate for curvature.

The combination of these two lemmas gives the following time decay estimate for curvature.

Time decay estimate: *Under the assumption of the theorem, we have* $T = +\infty$ *and*

$$|R_m(x,t)| \leq \frac{C}{1+t}, \ \text{on } M^2 \times [0,+\infty),$$

for some positive constant C.

Step 2. **Topology**

Consider the solution $g_{\alpha\bar{\beta}}(\cdot, t)$ of the Ricci flow. By using the local injectivity radius estimate of Cheng-Li-Yau (see also Cheeger-Gromov-Taylor),

$$inj(M, g_{\alpha\bar{\beta}}(\cdot, t)) \geq c_0(1+t)^{\frac{1}{2}}, \text{ for } t \in [0, +\infty),$$

where c_0 is a positive constant.

- $Ric(\cdot, t) \geq 0 \quad \Rightarrow \quad B_t(x_0, \frac{c_0}{2}(1+t)^{\frac{1}{2}}) \supset B_0(x_0, \frac{c_0}{2}(1+t)^{\frac{1}{2}})$

$$\left(\because \frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}} \leq 0 \right)$$

- $inj(M^2, g_{\alpha\bar{\beta}}(\cdot, t)) \geq c_0(1+t)^{\frac{1}{2}}$

$$\Rightarrow B_t(x_0, \frac{c_0}{2}(1+t)^{\frac{1}{2}}) \overset{diffeo.}{\cong} \text{ unit ball}$$

$$\Rightarrow \pi_p(M^2, x_0) = 0, \text{ for any } p \geq 1 \text{ and}$$

$$\pi_q(M^2, \infty) = 0, \text{ for } q = 1, 2.$$

$\Rightarrow$ (By the resolution of the generalized Poincare conjecture on four-manifolds by Freedman)

$$M^2 \overset{homeomorphic}{\cong} \mathbb{R}^4.$$

Thus we obtain

Proposition. *Under the assumption of the theorem, the manifold M^2 is homeomorphic to $\mathbb{R}^4$.*

Step 3. **Space decay estimate on curvature**

We now consider the question *how to get the curvature decay.*

$$\begin{cases} \bullet\textbf{Average Linear Decay}: \text{ bisect} > 0 \Rightarrow \\ \frac{1}{Vol(B(x_0,r))}\int_{B(x_0,r)} R(x)dx \leq \frac{C}{1+r}, \quad \forall r \geq 0, \\ \\ \bullet\textbf{Yau prediction}: \text{ bisect} > 0, \text{ maximal volume growth} \\ \overset{?}{\Rightarrow} \text{ curvature quadratic decay in certain average sense.} \end{cases}$$

Our idea:

• Evolve the metric by the Ricci flow to get a time decay estimate on curvature,

• Use the time decay estimate to derive a space decay estimate on curvature.

By definition

$$-\partial_\alpha\bar{\partial}_\beta \log \frac{\det(g_{\mu\bar{\nu}}(\cdot,t))}{\det(g_{\mu\bar{\nu}}(\cdot,0))} = R_{\alpha\bar{\beta}}(\cdot,t) - R_{\alpha\bar{\beta}}(\cdot,0),$$

after taking trace with the initial metric $g_{\alpha\bar{\beta}}(\cdot,0)$, we get

$$R(\cdot,0) = \triangle_o F(\cdot,t) + g^{\alpha\bar{\beta}}(\cdot,0)R_{\alpha\bar{\beta}}(\cdot,t)$$

where $\triangle_o$ is the Laplacian operator of the initial metric $g_{\alpha\bar{\beta}}(\cdot,0)$, and $F(\cdot,t) = \det(g_{\mu\bar{\nu}}(\cdot,t))/\det(g_{\mu\bar{\nu}}(\cdot,0))$.

Since

$$|g^{\alpha\bar{\beta}}(\cdot,0)R_{\alpha\bar{\beta}}(\cdot,t)| \le |R(\cdot,t)| \le \frac{Const}{1+t},$$

we have

$$\frac{\partial F(x,t)}{\partial t} = -R(x,t)$$

and then

$$|F(x,t)| \le Const. \log(1+t).$$

Thus

$$\int_{B_0(x_0,r)} \frac{R(x,0)}{d_0^2(x_0,x)} dx \le Const. \int_{B_0(x_0,r)} R(x,0) G_0(x_0,x) dx$$

$$(G_0 \text{ is a positive Green function w.r.t. } g_{\alpha\bar{\beta}}(\cdot,0))$$

$$\le Const. \int_{B_0(x_0,r)} (\triangle_o F(\cdot,t) + (\frac{Const.}{1+t})) G_0(x_0,x) dx$$

$$\le Const. (\log(1+t) + \frac{r^2}{t})$$

$$\left(G_0(x,x_0) \approx Const. \frac{d(x,x_0)^2}{Vol(B(x_0,d(x,x_0)))} \right).$$

Then by choosing $t = r^2$, we have

$$\int_{B_0(x_0,r)} \frac{R(x,0)}{d_0^2(x_0,x)} \le Const. \log(2+r), \text{ for all } r \ge 0.$$

Space decay estimate: *Under the assumption of the theorem, we have*

$$\int_{B(x,r)} \frac{R(y)}{d(x,y)^2} dy \le Const. \cdot \log(2+r), \quad \forall r \ge 0,$$

$$\left(\begin{array}{l} \textit{in particular,} \\ \frac{1}{Vol(B(x_0,r))} \int_{B(x_0,r)} R(x)dx \le Const. \cdot \frac{\log(2+r)}{r^2}, \quad \forall r \ge 0 \end{array} \right)$$

Consider the Poincare-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = Ric.$

Recall:

- $R(x) \le Const./(1+d(x_0,x))^{2+\varepsilon} \quad \Rightarrow u$ is bounded,
- $R(x) \le Const./(1+d(x_0,x))^{2} \quad \Rightarrow u$ is of logarithmic growth.

Fortunately, we can still use the average decay estimate

$$\int_{B(x,r)} \frac{R(y)}{d(x,y)^2} dy \le Const. \cdot \log(2+r)$$

to solve the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = Ric$ to get that u is of logarithmic growth.

Proposition: *Under the assumption of the theorem, we can find a strictly plurisubharmonic function u of logarithmic growth.*

Step 4. We basically follow the arguments of Mok.

• Use the strictly plurisubharmonic functions u as weight in the L^2-theory of $\bar{\partial}$-operator to get algebraically independent holomorphic functions f_1, f_2 of polynomial growth.

• By a Poincaré-Siegel argument and the primitive element theorem, the quotient field of holomorphic functions of polynomial growth is given by $R(M^2) = \mathbb{C}(f_1, f_2, g/h)$, for some holomorphic functions g, h of polynomial growth.

• The map $F = (f_1, f_2, g, h) : M \to \mathbb{C}^4$ defines a birational equivalence.

• Desingularize the map F and use Ramanujam's theorem.

#

The above argument is heavily depending on the Ramanujam theorem which is only valid in complex two-dimension. Nevertheless it can used to compactify complete noncompact Kähler manifolds of positive bisectional curvature and maximal volume growth for all dimensions.

Recently Chau and Tam had extended the above result to all dimensions by more direct method.

Theorem (Chau-Tam(2005))

Suppose: Kähler manifold M^n, $0 <$*bisect*$\leq Const.$, *and*

(i) $Vol(B(x_0, r)) \geq C_1 r^{2n}, \quad \forall r \geq 0.$

Then

$$M^n \overset{biholo.}{\cong} \mathbb{C}^n.$$

§6 Non-maximal Volume Growth Case

In this section we will consider the uniformization theorem without maximal volume growth assumption.

Theorem (To, *Duke Math. J.* (1991)) *Let M^n be a complete noncompact Kähler manifold of positive holomorphic bisectional curvature and suppose for some base point $x_0 \in M$ that there exist positive C_1, C_2 and p such that*

(i)'

$$\int_{B(x_0,r)} \frac{1}{(1+d(x_0,x))^{np}} dx \leq C_1 \log(r+2), \quad r > 0,$$

(ii)'

$$R(x) \leq \frac{C_2}{1+d(x,x_0)^p}, \quad \text{on } M^n,$$

(iii)

$$c_1(M^n)^n = \int_{M^n} Ric^n < +\infty.$$

Then M^n is quasi-projective. Moreover, if in addition the complex dimension $n = 2$ and the sectional curvature of M^2 is positive, then M^2 is biholomorphic to $\mathbb{C}^2$.

It is likely that the assumption (iii) is automatically satisfied for complete Kähler manifolds with positive sectional curvature. At least in the complex two-dimensional case, there holds the generalized Cohn-Vossen inequality

$$c_2(M^2) = \int_{M^2} \Theta \leq \chi(\mathbb{R}^4) < +\infty$$

where Θ is the Gauss-Bonnet-Chern integrand. In view of Miyaoka-Yau type inequality on the Chern numbers, it is reasonable to expect getting the finiteness of $c_1(M^2)^2$ from that of $c_2(M^2)$. Meanwhile in views of Demailly's holomorphic Morse inequality and the L^2-Riemann-Roch inequality of Nadel-Tsuji, the assumption (iii) is a natural condition for a complete Kähler manifold to be a quasi-projective manifold. The following result shows that the assumption (iii) alone is sufficient to give an affirmative answer.

Theorem(Chen-Zhu)

Let M^n be a complex $n-$dimensional complete non-compact Kähler manifold with bounded and positive sectional curvature. Suppose

$$c_1(M^n)^n = \int_{M^n} Ric^n < +\infty.$$

Then M^n is biholomorphic to a quasi-projective variety, and in case of complex dimension $n = 2$, M^2 is biholomorphic to $\mathbb{C}^2$.